Mathematical Time in Classical Mechanics
– An Epistemological Perspective –

Marcel BODEA S
Department of Philosophy,
Faculty of History and Philosophy,
Babeş-Bolyai University, Cluj

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Abstract: The aim of this essay is to analyze the signification of the mathematical definition of time within the framework of the mathematical methods of classical mechanics. The strictly mathematical definition of time, starting from the requirements of classical mechanics, is at a level of abstractization the specific formalism of which together with the physical intuitive sense of time taken from classical mechanics give a scientific prospect and signification to time considered in this double scientific context. The mathematical time of classical mechanics is a fundamental concept of physics. Starting from the description of its complexity included in its physical-mathematical definition, the study attempts to make certain enlightening explications, followed by an analysis based on an algebraic approach. In this essay, the "algebraic view" on time represents a reference point for the philosophical perspective on the time. The algebraic approach to time proves the complexity of the problems and brings into attention new aspects and at the same time difficulties to be revealed from other viewpoints. After we have become familiar with it, this approach offers new themes for philosophic reflection. We do not refer to a breach between the philosophical and abstract formal views in issues related to time, we only specify the existence of some real distinctions between the two approaches.

E-mail: bodeamarcel@hotmail.com

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Introduction

Time in this paper represents neither a pure sensibility of the epistemic subject, nor an eventual attribute of the ontic subject, and not even a philosophical category. What we propose in this study is an epistemological analysis on formal-algebraic bases of the mathematical definition of time in classical mechanics and we follow a consequence of this definition. Related to mathematical descriptions current intuitions, ontological presuppositions, and epistemological consequences are presented. Certain physical contents and certain mechanical significations of symbols within mathematical formalism being given, developing purely mathematically this formalism and obtaining formal results, the qualitative (mechanical) interpretation of these results must not conflict with the mathematical aspects and the initial physical presuppositions or with generally accepted physical presuppositions and representations. Otherwise, it must be recognized that the mathematical results in question lack physical significations and relevance (maybe only momentary). Sometimes, conferring to a certain mathematical formalism initial physical significations and developing it mathematically, mathematical results with unexpected physical signification can be obtained. In this case it is justified to search for some epistemological meanings which can be different from the strictly physical-mathematical ones.

Symmetries create patterns that help us organize our world conceptually. In mathematics, the idea of symmetry gives us a precise way to think about this subject. The uniformity of time is for example a symmetry. There is a theorem in classical mechanics formulated by Emmy Noether which makes possible one of the most general *associations* between spatial-temporal symmetries (the uniformity of time, the homogeneity and isotropy of space) and the most important principles of classical physics, the principles of conservation (the conservation of energy, of impulse, and of the kinetic moment). Epistemologically, we would much rather choose the term *association* than *correspondence* in order to avoid a possible "strong" both philosophical and scientific commitment such as: "the uniformity of time determines (causally) the conservation of energy". The epistemological interest of Noether's theorem is conferred by the relationship between the three very general and in their content very distinct "factors": 1.) *an ontological presupposition*; 2.) *a mathematical transformation of symmetry*; 3.) *a physical conservation principle*. Neither the *ontological presupposition* (the philosophical aspect), nor the *symmetry operation* (the mathematical aspect) are experimentally accessible, only the *conservation principle* (the physical aspect) of a certain physical extent. The analysis in this paper does not focus directly on the uniformity of time, but it will follow the same relations between the above factors in the variants: ontological and epistemological presuppositions, mathematic formalism, measurement operations (unambiguous numeric localization).

We start from the question: "For what is time? Who can readily and briefly explain this? [...] And, we understand, when we speak of it; we understand also, when we hear it spoken of by another. What then is time? If no one asks me, I know: if I wish to explain it to one that asketh, I know not..."[1] This passage from Saint Augustine's *Confessions* suggests, we believe, the relationship between intuition and conceptual analytic clarification. The multiple philosophical perspectives on time – the ways of questioning and the modalities of answering philosophically – are, it must be recognized, in most part far from the scientific discussions of the age. Here, the physical-mathematical perspective of classical mechanics will be considered. The scientific discussion of **time**, even if limited, proves the complexity of the problem. After it has become known – through conceptual clarification and epistemological clarification –, it offers new subjects for philosophical reflection. We do not refer to a breach between the philosophical and scientific views on issues related to *time*; we only wish to specify the existence of some real distinctions between the two approaches.

The mathematical definition of time in classical mechanics

In this analysis the algebraic notion of function is the central concept. Since the construction of the analysis is based on this notion, we are going to define it in a larger algebraic context. This will make possible more precise epistemological clarifications.[2]

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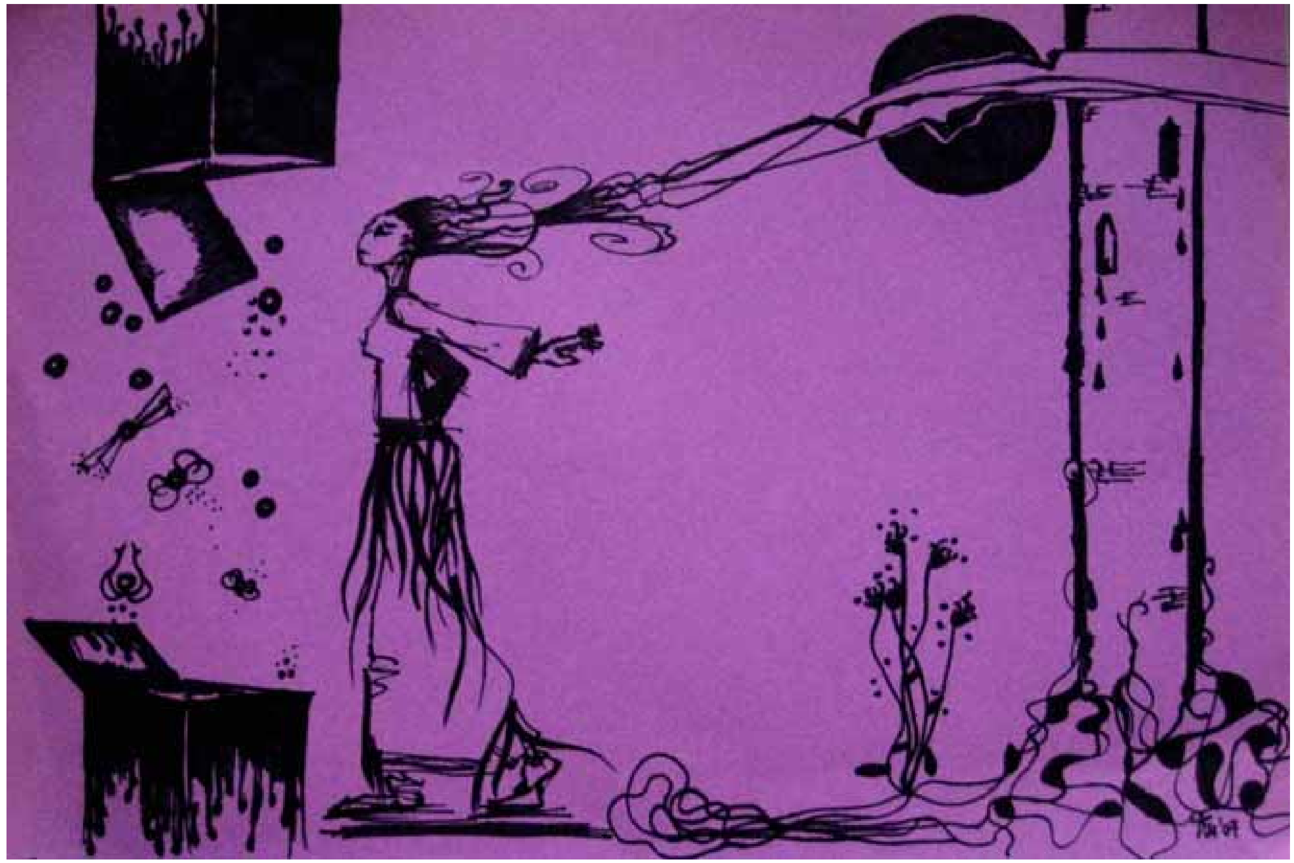
[1] Saint Augustine, *The Confessions of Saint Augustine*, transl. Edward Bouverie Pusey, http://www.gutenberg.org/ebooks/3296. Accessed on October 3, 2010.

[2] Ioan Purdea and Ioana Pop, "Cap. I. Relații" (Chapter 1. Relations), in *Algebră* (Algebra), (Zalău: GIL, 2003), 8, 13, 14.



Ana-Maria Călinescu, *MOV-ing*, Black marker on paper (460 × 305 mm.)

An ordered system $r = (A_i, A_j, R)$, where R is a subset of the Cartesian product $A_i \times A_j$ is called a binary relation between the elements of sets (A_i, A_j). For $(x_i, x_j) \in R$ the notation $x_i r x_j$ is used suggestively. In this way "relation" is extremely generally defined. Despite the mathematical generality of the definition, two interpretative observations are imposed on. Firstly: in order that two elements *may be* in relation, the sets they are thought to belong to must be "connected" by the Cartesian product. Secondly: in order that two elements *may be* in relation, the connection between the sets must be specified by means of a subset of the Cartesian product. We also presuppose that the relation *r* is given and that from its perspective we are interested in the sets A_i and A_j. Let the subset be $X \subseteq A_i$. In this context, subset $Y \subseteq A_j$, consisting of elements with which the elements of the subset X are related, is interesting. This will be defined in this way:

$$Y = r(X) = \{y \in A_j \mid \exists x \in A_i, xry\}$$

and it is called the *section of r in* A_j *based on X (subset)*. Further on, the situations where subset *X* consists of a single element $X = \{x\}$ are of additional interest. In this case the notation $r(\{x\}) = r\langle x \rangle$ is used and it is called *the section of r in* A_j *based on x (element)*. In the case of a relation $r = (A_i, A_j, R)$, section $r\langle x \rangle$ with $x \in A_i$ can be: empty; consiting of several elements; or consisting of a single element. Function relations are those in which section $|r\langle x \rangle| = 1$ consists of only one element. More precisely: a relation $f = (A, B, F)$ is called a function or an application or the function relation of *A* in *B* if for $\forall x \in A$ section $f\langle x \rangle$ consists of a single element denoted $f(x)$. *Applications, functions relations* and *functions* are very closely related. Terminologically speaking: *A* – domain of function; *B* – range of a function; $f(x)$ – the image of *x* under *f* (or the value of *f* when applied to *x*). What is essential in the former definition is the fact that function is a special relation where one and only one element corresponds to a given element. In common language it can be said that thus: all is distinct and clear; ambiguity is eliminated; in a certain sense, relativity, interpretation is eliminated. The difference between "the only possibility (x_1, y_1)" and "several possibilities (x_1, y_1), (x_1, y_2)" is suggestive. A function is a "well-behaved" relation. This means that while all functions are relations, not all relations are functions. Functions are a special sub-classification of relations. When we say that a function is "a well-behaved relation", we mean that, given a starting point, we know exactly where to go; given an *x*, we get only and exactly one *y*. Because of such "natural cognitive" reasons functions are important for the formal language of science, for scientific description. We confine ourselves only to these observations. In order to underline once again the "strong" restrictive requirement of the definition of function, let us mention by the way that in general the function relation permits for example (x_1, y_1) and (x_2, y_1).

What is time? We have chosen the following mathematical definition of time in classical mechanics: "**R** denotes the real number set and $\mathbf{R}^n$ the n-dimensional real linear space. A^n, the *n-dimensional affine space* differs from $\mathbf{R}^n$ by the fact that 'the origin of the coordinates is not fixed' in it. The group $\mathbf{R}^n$ functions in A^n as a *parallel displacement* group (translations).

$a \to a + \boldsymbol{b}\ (a \in A^n, \boldsymbol{b} \in \mathbf{R}^n, a + \boldsymbol{b} \in A^n)$

1.) THE UNIVERSE – a 4-dimensional affine space A^4. The points of the affine space A^4 are called *world points* or *events*.

2.) TIME – a linear application t: $\mathbf{R}^4 \to \mathbf{R}$ of the space of parallel displacements on 'the real time axis'.

The set of simultaneous events is a three-dimensional affine subspace of A^4 and it is called the *space of simultaneous events A^3*.

The kernel of a function f consists of the parallel displacements of A^4 which transform some (and therefore every) event into an event simultaneous with it. This kernel is a three-dimensional linear subspace $\mathbf{R}^3$ of the linear space $\mathbf{R}^4$."[1]

All the following epistemological considerations are based on the analysis of this mathematical text.

$\mathbf{R}$ is the real number set, which is an ordered set. Let us specify that set $\mathbf{R}^n$ with $n \geq 2$ is not an ordered set: plane, for example, is not an ordered set.[2] Suggestively, let the following be an intuitive temporal metaphor (time has not yet been defined) for $\mathbf{R}^n$: it is a world with "present", "past", and "future"; and, respectively, let the following be an intuitive spatial metaphor (space has not yet been defined) for $\mathbf{R}^n$: it is a world with "here" and "there". Comparatively, we suggest a metaphor for A^n: it is a world only with the temporal "sometimes" and the spatial "somewhere". Metaphorically speaking, we have used words of the everyday language: present, past, future, here, there, somewhere, sometimes, which inevitably fill mathematical formalism with certain senses, senses which through the nature of "formality" itself are arbitrary. In other words, mathematical formalism is neutral to any non-mathematical interpretation of its sense or signification: factual scientific, everyday, philosophical, theological, etc. interpretations. Somehow we must start, however, speaking about it. In what follows, we shall try to present the purely mathematical signification of formalism. Then a physical (mechanical) interpretation of this, and, finally, an epistemological interpretation exactly of this attempt to give a physical-mathematical signification will be proposed. We make the following statement, which, however, we do not elaborate on here: if there is no system of reference, an "origin of coordinates", then the "measuring operation" is not possible. We have specified above that A^n differs from $\mathbf{R}^n$ in the fact that the origin of coordinates is not fixed in it, therefore in A^n no measurements can be made. In a first formulation the translation $a \to a + \boldsymbol{b}$ *with* $a \in A^n$, $\boldsymbol{b} \in \mathbf{R}^n$ and $a + \boldsymbol{b} \in A^n$ means only that a point "has been removed" "from somewhere" "to somewhere else". Starting from here, the notion of *homogeneity* can be introduced in this line of definition: "Homogeneity means the preservation of 'something' by choosing arbitrarily some reference points." The choice (favouring) of a certain point named "*origin*" becomes in this way purely conventional, and it cannot belong to any qualitative specificity of the point in question. With the new notion we reconfigure and reread the above formal writing in this way: $a \to a \oplus \boldsymbol{b}\ (a \in A, \boldsymbol{b} \in \mathbf{R}, a \oplus \boldsymbol{b} \in A)$ mathematically means the homogeneity or uniformity of time; that is, $\mathbf{R}$ acts as a group of transformations which

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[1] This short mathematical explanation is based on: Vladimir Igorevich Arnold, "§2. Grupul galileian şi ecuațiile lui Newton" (The Galilean Group and Newton's Equations), in *Metodele matematice ale mecanicii clasice* (The Mathematical Methods of Classical Mechanics) (Bucureşti: Editura Ştiințifică şi Enciclopedică, 1980), 14–16.

[2] An algebraic order relation can be introduced, but this issue does not interest us here.

leaves the point-event a invariant in space A as a result to the (symmetry) transformation $\oplus$. In other words, the point-events a and $a \oplus \boldsymbol{b}$ are equivalent in the sense that $a \in A$ and $a \oplus \boldsymbol{b} \in A$ as a result of transformation $\oplus$. Observation: the homogeneity introduced above has an operational sense, that is, it results from the interpretation of an algebraic operation.

We shall refrain, in what follows, as far as possible from associating the homogeneity of time with some metaphysical content. But can tacitly assumed ontological presuppositions or commitments completely be ignored? Let a "complex operation" be a physical experiment in which some results of a measurement are recorded. If the experiment is repeated in identical conditions after an interval of time (temporal translation) the results of the measurement are the same. We usually say that the simple passing of time does not influence the results of a physical experiment. There is thus an "identity" exterior to the reading of the clock, a "conservation" which transcends the reading of an instrument of measurement; in other words, the translation does not affect the physical experiment. In the spirit of what we have said, in natural language the notion *symmetry* closely connected to the notion *operation* can be expressed in this way: we say that "something" is symmetrical in comparison with an operation if that "something" subjected to the operation remains identical. In the following paragraphs a certain "ontic identity" is presupposed in each of the formal constructions and their interpretations. The explanations will be given following each case.

In the mathematical and mechanical context our discussion is situated in, there are only two distinct possibilities for a point to be "removed".[1] Intuitively: to translate an object means to move it without rotating it and to rotate an object means to turn it around.

Any "removal" (displacement) consists of translations and rotations. We limit the mathematical discussion here only to $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{R}^3$. We affirm the following without explanation, as the statements are sufficiently evident intuitively: in $\mathbf{R}$ there is translation, but there is no rotation (only eventually as a marginal case); in $\mathbf{R}^2$ and $\mathbf{R}^3$ there is both translation and rotation. The translation of a point can always be represented unidimensionally and rotation bidimensionally. In other words, the "representation space" for translation is a unidimensional space $\mathbf{R}$, and for rotation a bidimensional space $\mathbf{R}^2$, even if the "description space" is the superior dimension.

We shall agree as a start that if we call the points in the space (UNIVERSE) A^4 "events", we shall call the points in $\mathbf{R}^4$ "events with a fixed origin". We shall agree to name the points or "the events with a fixed origin" from $\mathbf{R}^4$ "spatial-temporal events" or "physical events". A "spatial-temporal event" is thus a point of a quadridimensional linear space. Writing $\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$, it can be said that $\mathbf{R}^4$ is the Cartesian product of the three-dimensional linear space $\mathbf{R}^3$ with the unidimensional linear space $\mathbf{R}$. $\mathbf{R}^3$ and $\mathbf{R}$ are mathematical subspaces of $\mathbf{R}^4$. We have no intuitive interpretation for a mathematical point from $\mathbf{R}^4$, but we have an intuitive physical interpretation of the mathematical point as a physical event! This "intuitive-interpretative jump" is interesting from a

[1] Let be the *plane*. In fact, there are four types of symmetry in the plane: *Translation, Rotation, Reflection* (to reflect an object means to produce its mirror image) and *Glide Reflection* (a glide reflection combines a reflection with a translation along the direction of the mirror line).

philosophical point of view, but here it will be accepted as such.[1] We construct mathematically two applications: s: $\mathbf{R}^4 \rightarrow \mathbf{R}^3$ and, respectively, t: $\mathbf{R}^4 \rightarrow \mathbf{R}$. The ranges of application $\mathbf{R}^3$ and $\mathbf{R}$ have something familiar, they permit intuitive representations. In a first "intuitive approximation" they can be associated with physical space and time from classical mechanics. We shall agree to say that by writing $\mathbf{R}^4 = \mathbf{R}^3 \text{ X } \mathbf{R}$ the space $\mathbf{R}^4$ is "intuitively separable".[2] In other words, with reference to "dimensions", the quadridimensional *mathematical space* $\mathbf{R}^4$ can be represented by three dimensions associated intuitive-qualitatively to physical space (length, width, height) and by one dimension associated intuitive-qualitatively to physical time. It can be observed that through the former definitions time can be conceived independently from space in an absolute way (classical Newtonian mechanics), but associated with space with respect to representing a spatial-temporal event. The above definition on time covers a part of the common intuitions, but it is far from being familiar in this sense. It is a "working" definition, satisfactory only for the necessities of classical mechanics. We cannot represent a quadridimensional Euclidean space with all its four dimensions of the same "qualitative nature" in the sense of Euclidean dimensions from the three-dimensional physical space. The above considerations associate time with a *formal dimension* in the sense in which length, width, and height are dimensions. The qualitative association of the temporal dimension to the spatial dimensions, however, is not possible in our intuition. The "strong" hypothesis or presupposition from here, on the physical level of intuitive or physical-qualitative contents, is this: time is not space. This aspect seems to be especially important.

The signification of the application t: $\mathbf{R}^4 \rightarrow \mathbf{R}$ will be extremely useful to the later analysis as well.

In this introductory context let us discuss in brief the homogeneity of space and time. In the above construction of the time definition two quadridimensional linear *mathematical spaces* A^4 and $\mathbf{R}^4$ are used. The difference between these two spaces is an essential one: in space $\mathbf{R}^4$ we have an origin (of coordinates), while in space A^4 we have not. This latter space is called UNIVERSE. The lack of even conventionally privileged points is the maximal limit of the homogeneity (uniformity) of those points. Space $\mathbf{R}^4$ was separated into subspace $\mathbf{R}^3$ (subspace of the three-dimensional Euclidian space) and subspace $\mathbf{R}$ (subspace of the real time axis). In both spaces an origin of coordinates is fixed. The fixing of an origin privileges *conventionally* a point. The important question is raised: By introducing the "arbitrariness of the reference point" is the homogeneity of the points re-established? A basic relationship between the "arbitrariness of the reference point" and the "homogeneity of points" has been suggested above. This study, however, is interested in something situated behind the homogeneity (uniformity) of time, in a possible formal structure of time, more precisely, in a particular aspect: the formal structure of duration.[3]

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[1] This analysis has a limited epistemological aim, therefore, though we signal the existence of some philosophical problems, we accept them as such and pass over them.

[2] A possible metaphysical interpretative line of this "intuitive separability" is, for example, that of space and time as forms of the pure intuition of sensibility from the Kantian transcendental philosophy. The present analysis does not propose such an interpretation.

[3] There is an important characteristic of time without which "duration" for example cannot be defined in classical mechanics: *continuity* ("Time is continuous"). It will not be, however,

Translation as a relation[1]

Let be a clock. The clock is an intrinsic component of time; it gives the *measure* of time. In other words, important here is that we measure the time. Time t is thus the association of a (spatial-temporal) event "e" with a number representing the indication of a measuring instrument (clock). As a continuation of these statements the following affirmation seems to be banal: the clock measures durations. This analysis tries to suggest that this is not at all the case. From the temporal localization of an event to its duration, from "What is the time?" to "How much time has passed?", etc. there is a leap of "signification". These aspects are interesting rather from an epistemological, than a physical point of view. Algebra offers a formal instrument for epistemological analysis. Mechanics offers to formalism the frame for the content of sense and an image of the scientific relevance of epistemological significations.

Epistemological analysis does not begin in this point with an approach to the metaphysical nature of time, but with a linguistic question.[2] "A year ago..." or "Now, a year later..." are synonymous expressions in natural and scientific language. They give a natural-scientific signification to the arbitrariness of the origin of the temporal scale. Simply expressed: $t_1 = t_2 - \Delta$ or $t_2 = t_1 + \Delta$ or $t_2 - t_1 = \Delta$ (more concisely, with past/present/future, $t_2 = t_1 \pm \Delta, \Delta \geq 0$). Thus we shall opt naturally for *clock* **R** and for the operation addition corresponding to the reading of the clock.[3] We shall name *the reading of a clock: translation*. Simply and suggestively said, two moments t_1 and t_2 are related if the interval between them is considered.

Let be $x_i, x_j, \alpha \in$ **R**. We shall define translation T in **R** as an algebraic relation in this way:

$$x_i T x_j \Leftrightarrow \exists \alpha, x_j = \alpha + x_i$$

The relation thus defined is:

a.) reflexive law: $x_i T x_i \Leftrightarrow \exists 0, x_i = 0 + x_i$.

b.) transitive law:

$$x_i T x_j \wedge x_j T x_k \Leftrightarrow \exists \alpha, \beta \; x_j = \alpha x_i \wedge x_k = \beta x_j \Rightarrow$$
$$\Rightarrow x_k = \beta \alpha x_i \Rightarrow x_i T x_k$$

c.) symmetric law:

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discussed or effectively used in the present analysis, but it will be presupposed and accepted in order not to leave too abstract familiar notions completely devoid of intuitive contents.

[1] We shall consider implicitly known all the properties of the real number set, including the operations with real numbers, which make possible the discussion of **translation** as *relation* and as *operation*.

[2] "Let us consider first what we mean by **time**. What **is** time? It would be nice if we could find a good definition of time. [...] Perhaps we should say: 'Time is what happens when nothing else happens.' Which also does not get us very far. Maybe it is just as well if we face the fact that time is one of the things we probably cannot define (in dictionary sense) [...]. What really matters anyway is not how we **define** time, but how we measure it." – Richard P. Feynman, "Ch. Time and Distance, §5–2 Time", in Commemorative issue, *The Feynmman Lectures on Physics*, Feynman-Leighton-Sands, vol. I., (Massachusetts: Addison-Wesley Publishing Company, Reading, 1989), 5–1.

[3] The option for a certain operation is algebraically arbitrary.

$$x_i T x_j \Leftrightarrow x_j = \alpha + x_i \Leftrightarrow x_i = -\alpha + x_j \Rightarrow x_j T x_i .$$

Translation thus defined is an algebraic equivalence relation. Due to the properties of set **R** endowed with the operation of addition "+": $x_i T x_j, \forall x_i, x_j \in \mathbf{R}$. The immediate algebraic result is: $\mathbf{R} / \mathbf{T} = \{\{\mathbf{R}\}\}$, that is, the quotient set consists of a single equivalence class $\{\mathbf{R}\}$. This result, the "mathematic fact" that all the elements are in the same algebraic equivalence-translation relation with one another we interpret as *homogeneity*. Interpreting the algebraic translation relation as the reading of the clock for two temporal moments (considered explicitly or implicitly; for example: a physical measurement requires two explicit time moments; the zero hour, our age is an implicit moment for any historical dating), all the moments are equivalent, we speak in this sense of the homogeneity or uniformity of the time read ("recorded") on a clock. The homogeneity (uniformity) of time thus presented is, epistemologically speaking, a descriptive physical-mathematical interpretation on the level of a temporal reading of the phenomena (of experience). We wish to recall that time has been defined as a linear application t: $\mathbf{R}^4 \to \mathbf{R}$ of the space of parallel displacements on "the real time axis" (suggestively t : $\mathbf{R}^4 \to$ 🕓).

Two observations occur. In the construction of translation as relation: $x_i T x_j \Leftrightarrow \exists \alpha, x_j = \alpha + x_i$, the operational sense has been preserved through the presence of the symbol "+" signifying the addition of real numbers. Both the formal construction in this paragraph as well as its interpretation, though the main mathematical notions have an algebraic content, are in the traditional spirit of classical mechanics, nearer to mathematical analysis than algebra, and they cover our familiar intuitions related to mechanics. In comparison, the following paragraph is distanced from the classical approaches of mechanics by means of its accentuated algebraic, relational, and unoperational formalism, and by means of its interpretation. In this limited algebraic context we have avoided at the end of the analysis even a possible definition of duration as the measure of a set (of temporal projection).

The formal structure of duration as the section of a relation

Let us repeat and comment on the definition of time in a strictly algebraic perspective. In this paragraph, the reading of a clock is only a simple reading "f(x)" of a value (more generally, of an element) from the range of a function. Contrary to the previous paragraph, the reading of a clock has not even the intuitive covering of translation (the hands of a clock are moving[1]). The former paragraph has the role of the "abstract-intuitive" first term, if we can specify it so, of a comparison which facilitates the better understanding of the second, "abstract, non-intuitive" term; a term on which the present analysis mainly focuses. In the definition time t and the real axis of time **R** are clearly distinguished. Let us clarify this distinction. The distinction between *time* and *the real time axis* appears very clearly: *time* is a linear application (function), which associates, relates a space–time event with a point on the real axis **R** called in this case the real time axis. We agree to name the points of the real time axis "moments". It is understood that *time* is not identical with its moments. If the moments are real numbers, the properties of

[1] Let us say, by the way, that even if a clock is generally a round and the hands as physical objects make a rotational movement, that which can be read on the clock is a translation; in this sense hourglasses connect better the physical "image" to the physical signification.

real numbers are associated with the *time*. Further on, the topological structure of the real number set will not be interesting for us. Evidently, *time* is not identical with the events to which it associates moments either. *Time*, in other words, is the association of an event with "something measurable", with "a quantity", mathematically with a number (representing, mechanically speaking, the indication of a measuring instrument: the clock). In other words, in the definition of time the presence of a clock (the range of function **R**) is included. Let us elaborate a bit further on these considerations from a mechanical point of view. The functional correspondence allows a measurable property to be associated unequivocally to a physical event, that is, by means of a measuring instrument. We emphasize once again that the "clock" in mechanics is more than the mathematical set **R**. The mechanical temporal localization is a real number accompanied by a physical unit of measure. Time thus is a **physical quantity**. In order to underline the direct import of the analysis of this subject we quote the following characterization by Albert Einstein: "Under these conditions we understand by the 'time' of an event the reading (position of the hands) of that one of these clocks which is in the immediate vicinity (in space) of the event. In this manner a time-value is associated with every event which is essentially capable of observation."[1] In order to be able to speak, however, about the time of a phenomenon a unit of measure, implicitly, a "duration" must be defined.

For the sake of epistemological strictness we analyze at this point a distinction. We say that **R** is a mathematical clock and not a physical clock. For the sake of epistemological clarity we leave this distinction unclear almost throughout the entire discussion. The particular aim of this analysis, beyond establishing the relationship between the mathematical, philosophical (ontological-epistemological), mechanical (scientific) languages, is to define the formal structure (here in an algebraic sense) of duration. It will be observed that presuppositions complementary to the formal structure are necessary to define indeed duration.

We resume the analysis of mathematical formalism. Since time is a function, reading the clock associates to an event one and only one *moment*. Thus an event has no "duration" in the common sense of the term! Let us analyze now the absence of the clock. In the mechanical interpretation of mathematical formalism introduced above, **if the clock is absent, we do not have time**!

We are going to introduce and to discuss now the concept of *simultaneity*.[2]

[1] Albert Einstein, "Chapter 8. On the Idea of Time in Physics", in *Relativity: The Special and General Theory*, transl. Robert W. Lawson (New York: Henry Holt and Company, 1920), Bartleby.com. www.bartleby.com/173/. [Accessed on October 5, 2010.]

[2] The algebraic definition of simultaneity is abstract. Simultaneity in physics has other requirements: "[...] with all physical statements in which the conception 'simultaneous' plays a part. The concept does not exist for the physicist until he has the possibility of discovering whether or not it is fulfilled in an actual case. We thus require a definition of simultaneity such that this definition supplies us with the method by means of which, in the present case, he can decide by experiment whether [...] [the two events; our note, M. B.] occurred simultaneously. As long as this requirement is not satisfied, I allow myself to be deceived as a physicist (and of course the same applies if I am not a physicist), when I imagine that I am able to attach a meaning to the statement of simultaneity." – Ibid.

For the epistemological clarity of the analysis we present some necessary algebraic complements. For simplicity's sake, we define the kernel of a function with one variable.

Let $f : A \to B$ be a function. The relation between the elements of *A*, the graph of which is marked ker *f* and defined by $\ker f = \{(x_1, x_2) \in A \times A \mid f(x_1) = f(x_2)\}$ is called the *kernel* of *f*.[1]

Starting from this algebraic definition, we construct the notion of algebraic simultaneity in the present case. We rewrite in order to be more suggestive $f : E \to R$ where we agree to name the elements $e \in E$ algebraic events and the range of function *R* algebraic clock. Then, for two events, the kernel of function *f* is the definition of their algebraic simultaneity.[2] Algebraic simultaneity as a formal reference for the analysis of temporal simultaneity requires a separate discussion, all the more as algebraic simultaneity defines a relation of equivalence and implicitly equivalence classes and a quotient set, but it will be discussed here only to such an extent that is strictly necessary to the present analysis.[3] For simultaneous elements a "space of simultaneous elements" can be defined, which can be considered an autonomous space. It is, in fact, something which mechanics usually calls "space independent of time".

It has been stated above that, according to the mechanical interpretation of mathematic formalism, if the clock is absent, we have no time. We add this affirmation some other important observations. Physics in general, mechanics in particular describe physical phenomena by means of physical quantities. A physical quantity has a unit of measure which implicitly presupposes an operation of measuring. Thus only the applications among sets $\mathbf{R}^n$ ($n = \overline{1,4}$) have a physical (mechanical) sense. Now, we shall discuss the possible epistemological interpretations and the possible ontological commitments of some applications in which not everywhere are sets $\mathbf{R}^n$.

Let us consider the following application:

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1 Purdea and Pop, "Cap. I. Relații", 22.

2 Suggestively, for n simultaneous elements: $\ker f = \left\{ (e_1, e_2, ..., e_n) \in \underbrace{E \times E \times ... \times E}_{\text{de n-ori}} \mid f(e_1) = f(e_2) = ... = f(e_n) \right\}$.

3 For the formal definition, let *X* and *Y* be sets and let *f* be a function from *X* to *Y*. Elements x_1 and x_2 of *X* are *equivalent* if $f(x_1)$ and $f(x_2)$ are equal, i.e. are the same element of *Y*. The kernel of *f* is the equivalence relation thus defined. The kernel, in the equivalence-relation sense, may be denoted "$=_f$" (or a variation) and may be defined symbolically as $(x =_f y) :\Leftrightarrow (f(x) = f(y))$. Like any equivalence relation, the kernel can be modded out to form a quotient set, and the quotient set is the partition: $\left\{ \left\{ \tilde{x} \in X : f(x) = f\left(\tilde{x}\right) \right\} : \tilde{x} \in X \right\}$. This quotient set $X/=_f$ is called the *coimage* of the function *f*, and denoted "coim *f*" (or a variation). The coimage is naturally isomorphic (in the set-theoretic sense of a bijection) to the image, im *f*; specifically, the equivalence class of *x* in *X* (which is an element of coim *f*) corresponds to $f(x)$ in Y (which is an element of im *f*). Like any binary relation, the kernel of a function may be thought of as a subset of the Cartesian product *X* × *X*. In this guise, the kernel may be denoted "ker *f*" (or a variation) and may be defined symbolically as: $\ker f = \{(x, x') \mid f(x) = f(x')\}$.

$$l_t : A^4 \rightarrow \boldsymbol{R}$$

What signification could this function have? Written in this way, none. Any interpretation would be a purely metaphysical speculation, something extremely general, such as: "There is time.", etc. We shall construct the application beforehand:

$$r_f : A^4 \rightarrow \boldsymbol{R^4}$$

which means the "representation of a point of the universe as a physical event (physical representation)". We shall use now the definition of time t: $\boldsymbol{R}^4 \rightarrow \boldsymbol{R}$ and only now we define the application l_t :

$$l_t : A^4 \rightarrow \boldsymbol{R}, \qquad l_t = t(r_f).$$

Thus l_t receives a meaning: it represents the **temporal localization** of an event from the universe A^4. We wish to emphasize that: an event has one and only one "instantaneous" temporal localization (one moment associated).

Similarly constructed

$$l_s : A^4 \rightarrow \boldsymbol{R}^3$$

application l_s represents the **spatial localization** of an event from the universe A^4. Let us specify also for spatial localization what we have mentioned with regard to temporal localization: being a function, an event has no "extent"! Let us mention that the temporal and spatial localizations of events of universe require "points of origin" only in the range of functions which we agree to call "spaces of localization". We have agreed to say that by the writing $\mathbf{R}^4 = \mathbf{R}^3 \text{ X } \mathbf{R}$ space $\mathbf{R}^4$ is "intuitively separable". We shall make, at this point, an additional convention. From the definitions of spatial and temporal localizations and from the writing $\mathbf{R}^4 = \mathbf{R}^3 \text{ X } \mathbf{R}$, we shall agree to consider an event (particularly a physical event) completely localized spatially and temporally. We consider that any event from the Universe A^4 is only localized spatially and temporally, but it is *localized* ***both*** *spatially* ***and*** *temporally*! The necessity of a double localization is essential![1] Moreover, it is presupposed that temporal and spatial localizations are mutually independent (but they can be associated in the spatial-temporal localization of an event).

At the same time, we can observe that we cannot talk about the events of the universe "directly", only "mediately", by means of physical events. Thus, an element of universe $e \in A^4$ is localized at the moment t_1 etc. only if a physical event can be associated to it: $e \rightarrow f(e) \in \boldsymbol{R}^4$. Let us interpret now mathematical formalism as it is given.

$r_f : A^4 \rightarrow \boldsymbol{R^4}$ mathematically is a function, which means that one and only one physical event corresponds to an event. This mathematical requirement is very important for science and it leads to consistent epistemological interpretations. In the present study we limit ourselves only to the analysis related to time. In what follows the "temporal projection" or "formal structure" of duration will be algebraically introduced.

The *function* l_t: $A^4 \rightarrow \boldsymbol{R}$ means that for an event we have one and only one temporal localization (evidently that also for a physical event: $t : \boldsymbol{R}^4 \rightarrow \boldsymbol{R}$). This fact is

[1] We do not discuss in this paper, though it is very interesting and important, that the reduction to a single localization makes impossible *classical mechanics* as it is constructed. (We have a series of arguments to this effect, but the fact that we do not preset them here reduces this note only to a simple statement.)

something familiar to us and we shall specify it more suggestively related to space.[1] In classical mechanics a body occupies, at a given moment, one place in space and it is not in two different places: here and farther on.[2] As we have seen, two different events can however be simultaneous. Now an aspect follows which will be more difficult to present and to interpret. Resorting however to mathematical results (which we shall not analyze in their mathematical details) and to an intuitive support which is based on our everyday as well as scientific experience, we hope we shall clarify sufficiently the problem.

Let be an event e to which the temporal localization $l_t(e) = t_1$ is associated. Mathematically this representation remains immutable: "e remains e" in the "same t_1 which remains t_1". Our *physical experience*, however, makes us say sometimes that we localize an event e in the moment t_1 and the same event e in the moment t_2, $t_1 \neq t_2$. Remaining consistent with the mathematical signification of the function and accepting that set R is mathematically continuous, we shall give a formal framework, a *structure* to (time) "duration".[3] Let therefore be: $l_t(e) = t_1, l_t(e) = t_2, ..., l_t(e) = t_n$. Of course, "global writing": "*function* l_t: $A^4 \to \boldsymbol{R}$ and $l_t(e) = t_1, l_t(e) = t_2, ..., l_t(e) = t_n$" represents thus a mathematical inconsistency! Let us specify an important aspect. If, in particular, only temporal localization $l_t(e) = t_1$ or the only spatial $l_s(e) = s_1$ can lead, in the above sense, to mathematical inconsistencies, the double localization $l_{st}(e) = (s_1, t_1)$ **allows** for formal (and implicitly ontological and epistemological) distinctions which are relevant because of their consequences. First of all, it is a significant formal-mathematical aspect that the above signalled inconsistencies disappear. In what follows we shall not clarify all the detail of the kind event→physical event, etc., we shall only give the symbols and their interpretations strictly necessary for the analysis.

Let therefore be:

$$l_{st} : A^4 \to \mathbf{R}^3 \times \mathbf{R}, \ \ l_{st}(e) = (s_i, t_i),$$

with the interpretation: "An event of Universe is localized if and only if it is localized spatio-temporally." We have seen that we talk about the localization of an event only through the mediation of a physical event. Similarly, we talk about particular temporal or spatial localizations beyond their strict definitions only through the mediation of the double spatio-temporal localization. Only this makes possible science (here classical mechanics).

The following analysis can be particularized, of course, on physical events, but on the abstract level it will be presented on, it is, without loosing its strictness, more suggestive with reference to the events. We have all the above definitions and constructions. In this context, we wish to "save mathematical description in the language of functions (in other words, not to renounce functions)". The solution we propose

[1] We have used the word *body*, which intuitively (or in the Kantian metaphysical sense) has an extent which we abstract here, the temporal and spatial localizations of events are punctual; "duration" and "extent" must be constructed.

[2] We wish to emphasize that the epistemological analysis with reference to classical mechanics does not in any way refer to an explicit analysis of some similar questions of quantum mechanics. We only mention that, mainly, from most perspectives, things are the same in quantum mechanics too as far as we operate mathematically with functions.

[3] Mathematical continuity plays no role in the following considerations, it is only the formal expression of an intuitively presupposed content.

comes from the direction of algebra (it is suggested by the algebra). We consider the algebraic notions *relation of equivalence, equivalence class, quotient set, factor set, representative of an equivalence class,* and *canonical projection map p* must be familiar, therefore we do not dwell on them. The statements we shall make can be formalized and sustained algebraically. They are however evident enough, therefore we do not burden the analysis with a formal complementary sustaining.[1] For the sake of suggestiveness and clarity, we shall draw an intuitive analogy.[2]

Let us present, to this effect, the formal-algebraic reference of the analogy very briefly: the rational number set *Q*, as well as the fraction set $\frac{m}{n}, m, n \in \mathbf{Z}$. Algebraically, a relation of equivalence is a relation between the elements of a set and it determines classes of equivalence. A certain relation of equivalence determines certain classes of equivalence.[3] The relation of equality is an "algebraic relation" of equivalence.[4] As compared to this relation of equivalence for example $\frac{2}{3}$ and $\frac{10}{15}$, respectively $\frac{3}{5}$ and $\frac{9}{15}$ belong to the same class of equivalence $C_{\frac{2}{3}} = \left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\right\}$, respectively $C_{\frac{3}{5}} = \left\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \ldots\right\}$.[5]

Now we construct the epistemological analogy, developing, on the one hand, the mathematical aspects, on the other hand, an interpretation of these. We repeat the previous writing "*function* l_t: $A^4 \to \mathbf{R}$ and $l_t(e) = t_1, l_t(e) = t_2, \ldots, l_t(e) = t_n$" which, as we have observed, led to a mathematical inconsistency. We shall denote the event $e \in A_4$, which we localize in the moment $t_i \in \mathbf{R}$, $e(t_i)$. We emphasize that this is only a rewriting which does not solve the difficulties of the problem. "Function" is not saved.

Let be the set of events A^4. There is an intuitive-empirical basis for the following affirmation, which formally can be made also without an empirical legitimation: an event *e* can be localized temporally differently. Thus $e(t_1) \in A^4$ and $e(t_2) \in A^4$ can represent one and the same event localized temporally in the two different moments. That which we have proposed to do requires more than one localization. We have considered above that an event (a physical event) is completely localized spatio-temporally. We shall write this simply: $e(s_i, t_i)$ ($e(s_i, t_i) \in A^4$). Before continuing let us take a general look on some philosophical aspects related to certain *ontological*

[1] When we discussed translation as a relation we gave a simple example of algebraic formalization.

[2] "Intuitive analogy", of course, does not substitute mathematical description.

[3] An intuitive example: the "similarity of triangles" as a relation of equivalence determines certain classes of equivalence in the set of triangles, while the "equality of areas" as a relation of equivalence determines other classes of equivalence in the same set. The "structure" of the triangles' set as compared to the similarity and equality of areas is different.

[4] An algebraic presentation on the relations and classes of equivalence can be found in Purdea and Pop, "Cap. I., §1.8. Relații de echivalență şi partiții" (Chapter I. §1.8. Relations of Equivalence and Partitions), in *Algebră*, 43–48.

[5] For the sake of simplicity and briefness I reduced $m, n \in \mathbf{Z}$ to $m, n \in \mathbf{Z}_{+}^{*}$.

presuppositions and their epistemological consequences. The first presupposition: there are events of universe which can be localized spatio-temporally (even if the spatial and temporal localizations of the event are not clarified). This allows for the formalization of events through functions of spatio-temporal localizations. In order not to overload the discussion formally (with restrictions of function, etc.), we shall presuppose moreover that all the events from the universe A^4 can be temporally localized (evidently spatially as well). Of course, philosophically, and especially metaphysically speaking, it is an extremely restrictive presupposition. But, taking into account that the analysis is aimed at time in classical mechanics, the presupposition is acceptable. The second presupposition: all the events which can be spatio-temporally localized are in the universe A^4. The third presupposition: any spatio-temporal localization is possible; in other words: any temporal localization can be associated to any spatial localization, and *vice versa*. We shall express formally the last two presuppositions in this way: let be I a set of indices and $\forall i, j \in I, e(s_i, t_j) \in A^4$.[1]

We mention that the present analysis is different from the analysis based on the *ontological presupposition* that in any moment, at any reading of the clock there is, at least theoretically, an event of universe which can be associated to the clock (the instrument of measure). Thus a formula: each time we look at the clock, t_i, **there is** an event, e_i, in the universe which can be localized $l_t(e_i) = t_i$. This presupposition legitimates the construction (definition) of such a function: c_e: $\boldsymbol{R} \to A^4$, $c_e(t_i) = e_i$ which we agree to name the "clock–event function"; this function is the mathematical expression of the ontological presupposition. The "clock–event function" is not the reverse of the function l_t. Such a presupposition is in the background of the familiar temporal dependence formulated suggestively by means of the dependence on parameter t: $e(t)$. Even in this discussion the *dependence* on time of certain events does not mean more than a descriptive dependence, that is, a simple association to the numbers of a clock (temporal localization). Clarifying on the level of current language the "flowing of time" (which in the proposed discussion will be understood as "duration"), does not affect at all the physical event. Scientifically speaking, the physical characterization of an event at a given moment is due to some factors different from what we call, in a way or another, *time*.

The analysis proposed here is a different discussion. We continue our train of thoughts by presenting the consequences of the presupposition that all the events can be localized spatio-temporally. This ontological presupposition and the requirement of spatio-temporal localization as a functional descriptive relation $e(s_i, t_i)$ leads to a formal requirement which we have interpreted as an *epistemological consequence*. For the sake of simplicity, we comment on a particular example: the events $e(s_1, t_1)$, $e(s_1, t_2)$, $e(s_2, t_1)$ are all different from one another. The epistemological consequence is the following: the operation of measurement (physical measurement, for example) differentiates between the events! There is, however, a "philosophical tension" here. On the basis of everyday and scientific experience we affirm naturally, without being too much of a Heraclitan, that certain events do not change, however, in time, at least for a

[1] We do not discuss the nature of the set of indices *I*. By this the strictness of the analysis does not suffer.

time. Thus, tacitly, a partial Parmenidean position is accepted on an ontic level. Can such a position be "saved" ontically? The following algebraic analysis and the interpretation of its results allows for the concomitant saving of both the mentioned ontic requirement and the formal description through functions. For us, however, the interpretative consequences on time are interesting.

Let be the spatio-temporal localization l_{st}.[1] Associated to this localization, we construct the following relation "~" between events localized (only) temporally:

$$e(t_i) \sim e(t_j) \Leftrightarrow (s_i, t_i) \wedge (s_i, t_j)$$

where $(s_i, t_i) \wedge (s_i, t_j)$ means: the distinct events localized spatio-temporally $e(s_i, t_i)$ **and** $e(s_i, t_j)$ *considered together*. With two events thus localized through definition the above relation "~" is constructed. The sign "$\wedge$ (and)" is the symbol of a special relation "considered together".[2] Observation: A possible basis for construction $(s_i, t_i) = (s_i, t_j)$ considered strictly algebraically as equality between two ordered pairs of the mentioned Cartesian product, leads directly to: $(s_i, t_i) = (s_i, t_j) \Rightarrow t_i = t_j$, a result which raises no epistemological interest. Before continuing the algebraic analysis let us clarify the sense of the relation $e(t_i) \sim e(t_j)$. Any temporally localized event is, according to the ontological presupposition, also localized spatially. Consequently, $e(t_i)$ is localized spatially, and, for simplicity's sake, we denote $e(s_i, t_i)$ the double localization. According to the second and the third presuppositions, there is in A^4 the event $e(s_i, t_j)$. We give now the following analogy which is purely suggestive and permits expression in the natural language, with some intuitive support: $\frac{3}{5} = \frac{6}{10} \Leftrightarrow \frac{3}{5} = \frac{3 \cdot \overset{1}{\cancel{2}}}{5 \cdot \underset{1}{\cancel{2}}}$, expressed otherwise, $\frac{3}{5}$ and $\frac{6}{10}$ are in a relation "~" (here meaning the equality-*equivalence* of fractions) because a simplification is possible.[3] We have seen that $\frac{3}{5}$ and $\frac{6}{10}$ belong to the same equivalence class: $C_{\frac{3}{5}} = \left\{ \frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \ldots \right\}$. But an equivalence class is considered a set consisting of several *elements*. Then, its elements are distinct from one another,

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[1] We no longer explain the associated formal writings, but we concentrate on the main aspects aimed at in the analysis.

[2] This special relation of "being together ($\wedge$)" can be over-formalized in this way: 1. $a \wedge b \Leftrightarrow a, b \in X$; 2. $a \wedge b \wedge c \Leftrightarrow a, b, c \in X$. We no longer develop formally these questions. The sense of the relation "$\wedge$" is given by 1. and 2. It also has an intuitive basis.

[3] For a strictly general algebraic discussion of fractions we refer to: Purdea and Pop, "Cap. IV. Semigrupuri şi inele de fracţii" (Chapter 4. Semigroups and Rings of Fractions), in *Algebră*, 169–180. In the present analogy we reduced the algebraic generality without "reducing" the mathematical correctness. Similarly, the interpretation of mathematical formalism is epistemologically oriented, without however, the given interpretation "clashing" with the algebraic interpretation.

they are *different*, such as the elements $\frac{3}{5}$ and $\frac{6}{10}$, *as elements*, different in $C_{\frac{3}{5}}$. Then what does the writing $\frac{3}{5} = \frac{6}{10}$ mean? The sign "=" is not a proper equality in the sense of "identity". It refers to something else here: to the elements of a class of equivalence which have the same *numeric value* of rational number. In this sense the "rational"-quantitative aspect is preserved and reading becomes unique in the decimal representation: the "rational number 0.6" for any of the representatives of the equivalence class $C_{\frac{3}{5}}$, for example. Let us continue with the observation that all the fractions of an equivalence class represent one and the same "mathematical object": a certain rational number *q* (for which we can give a unique representation in decimal form). In the present case, moreover, any object-rational number has a class of fractions which represents it. In this sense therefore, it can be written that: $\frac{3}{5} = \frac{6}{10} \Leftrightarrow \frac{3}{5} \sim \frac{6}{10}$.

Let us repeat, **suggestively**, the analogy. We have seen that the sign "$\wedge$ (and)" is the symbol of a general relation of "considering together" two events with certain similarities in the spatio-temporal localization. This will lead to the elements of a class of equivalence, which have the same *ontic value* of "determined ontic event". In this sense the *referential*-"ontic" aspect is preserved and the reading becomes unique in the ontic representation: "the determined ontic event *e*" which is the same for any of the representatives of an equivalence class which will be introduced later. More clearly, the idea is that the spatio-temporal localization (the operation of measurement) introduces the epistemic difference between the events: two events are considered different if at least one of the localizations is different. We resume the discussion only with respect to temporal localization (with reference only to spatial localization). We introduce two definitions. The first: "An event is spatially *ontically determined* if a spatial localization is associated to it." The second: "We say about two events which have the same spatial localization that they have the same *ontic value* of 'spatially determined ontic event'." With these definitions the sign "$\wedge$" from $(s_i, t_i) \wedge (s_i, t_j)$ has a well specified meaning. Through the *analogy* with $\frac{3}{5} = \frac{6}{10} \Leftrightarrow \frac{3}{5} \sim \frac{6}{10}$ we introduce: $e(t_i) \sim e(t_j) \Leftrightarrow (s_i, t_i) \wedge (s_i, t_j)$. Also through analogy, we say that $e(t_i)$ and $e(t_j)$ are in a "~" relation, because we can "simplify" space in the double localization, we can "simplify" it in the sense that being the same spatial localization, we can disregard it and "limit" ourselves to the temporal localization. In order to confer more clarity to the language and to give an additional support to the analogy in the way we express ourselves, we elaborate on what we have said by introducing an additional distinction. We say about two events that they are completely distinct from an epistemological and an ontic point of view if both their spatial and their temporal localizations differ. If the spatial localizations are the same and we are directly and with respect to consequences interested in temporal localizations, we make a simplification of localization in the sense that we renounce spatial localization (the same for all the considered events) which we presuppose only implicitly, and we speak only of the temporally localized events $e(t_i)$. The analogy was only a suggestive way of speaking about an abstraction.

The relation "~" defined above is:

a) reflexive law: $(s_i,t_i)\wedge(s_i,t_i)\Leftrightarrow e(t_i)\sim e(t_i)$.

b) transitive law: $(s_i,t_i)\wedge(s_i,t_j)$ and $(s_i,t_j)\wedge(s_i,s_k) \Rightarrow \Rightarrow(s_i,t_i)\wedge(s_i,t_k)$, which, according to the definition of relation, returns to writing: $e(t_i)\sim e(t_j)$ and $e(t_j)\sim e(t_k) \Rightarrow e(t_i)\sim e(t_k)$.

c) symmetric law: $(s_i,t_i)=(s_i,t_j)\Rightarrow(s_i,t_j)=(s_i,t_i)$, namely $e(t_i)\sim e(t_j)\Rightarrow e(t_j)\sim e(t_i)$.[1]

Thus it has been demonstrated that relation is an algebraic relation of equivalence.

Let be, by analogy, the set of events A^4 and, respectively, the set of events localized only temporarily. Algebraically, a relation of equivalence is a relation between the elements of a set and it determines classes of equivalence. The introduced relation "~" is an "algebraic relation" of equivalence between events localized temporally in the set universe A^4. As compared to this relation of equivalence $e(t_i)\sim e(t_j)\Leftrightarrow(s_i,t_i)\wedge(s_i,t_j)$, $e(t_i)$ and $e(t_j)$ belong to the same equivalence class, respectively $C_{e(t_i)}=\{e(t_i),e(t_j),.....\}$. They are epistemic events different with respect to temporal localization, but with the same *ontic value* of "spatially determined ontic event".

Let us specify that any element of a class of equivalence can be conventionally introduced as a representative of that equivalence class. Thus, in the above denotation: $C_{e(t_i)}=C_{e(t_j)}$. In such equivalence classes any temporally localized event can replace any other event of the equivalence class to which it belongs (can be a representative of an equivalence class). We can speak about a "spatial ontic similarity (identity)" in this case: for example $e(t_i)$ and $e(t_j)$ are identical ontic-spatial events. From an ontological point of view, the events of an equivalence class can be considered a partial ontic equivalence form to express the existence of a certain ontological (ontic) identity. The events of the equivalence class can be substituted for one another, they can alternate in an epistemological context without the ontic identity of reference being affected. To introduce such an interpretation of algebraic formalism is not at all a strictly mathematical interest. Its necessity may appear however from the perspective of some philosophical investigations regarding concepts, the relationship between them, the relationship between formal languages and fundamental scientific concepts, in particular the relation of a language to the objects of another language. Thus, such questions as: "What kind of sentences are the equations of mathematical physics?" or "How does mathematics describe the world of fact?" are philosophical problems.

In this construction, given the Universe-set A^4 and the equivalence relation ~ on A^4, the *equivalence class* of an element $e(t_i)$ in A^4 is the subset of all elements in A^4 which are *equivalent* to $e(t_i)$: $\{e(t_j)\in A^4 \mid e(t_j)\sim e(t_i)\}$. The notion "equivalence class" was useful here as it introduced new sets of events with a certain specificity,

[1] The formal signification for "$\wedge$" is the one we have given in a former footnote.

selected from the general set of the events of the Universe A^4. This procedure can be interpreted as an act of "temporal selection", by means of the equivalence relation, of the events from the Universe-set A^4.

Now the notion of "temporal projection" is going to be introduced which we agree to name the "formal structure of duration" in the algebraic-epistemological sense of the previous analysis. Its definition will reveal that the *temporal projection* eliminates the intuition of the flow of time.

Let be quotient set of A^4 by ~ of the equivalence classes induced by the relationship of equivalence defined above. An equivalence class is representative then, without the introduction of additional denotations, thus:

$$\sim\langle e(t_i)\rangle = \{e(t_j)\in A^4 \mid e(t_i)\sim e(t_j)\}.$$

The $\sim\langle e(t_i)\rangle$ can be used to denote that we mean the equivalence class of the element $e(t_i)$ specifically with respect to the equivalence relation ~. This is said to be the ~ - equivalence class of $e(t_i)$.

With these denotations, quotient set of A^4 by ~ is represented:

$$A^4/\!\sim\, = \{\sim\langle e(t_i)\rangle \mid e(t_i)\in A^4\}.$$

For any equivalence relation, there is a *canonical projection map p* from *X* to *X*/~ given by $p(x)=\langle x\rangle$. This map is always surjective. Let be canonical projection map *p*: $p: A^4 \to A^4/\!\sim$, $p(e(t_i)) = \sim\langle e(t_i)\rangle$. Written in the form of a relation we have: $(A^4, A^4/\!\sim, p)$. In this way, an equivalence class of the temporally localized event $e(t_i)$ is the section of *p* in $A^4/\!\sim$ based on $e(t_i)\in A^4$.

Let now be relation

$$p_r t = (A^4/\!\sim, \mathbf{R}, P_r t), \text{ with } P_r \subseteq A^4/\!\sim \times \mathbf{R}.$$

In order not to fill the analysis with formal elaborations, the relation will be, anyway rigorously, *explained* in words: each equivalence class is associated to each of the temporal localizations of the events which belong to it. For example: $(e(t_i), t_i)\in A^4/\!\sim\times\mathbf{R}, (e(t_j), t_j)\in A^4/\!\sim\times\mathbf{R}$ etc.

We can now define "temporal projection". "***Temporal projection*** *is a section of* p_r *in R based on* $\sim\langle e(t_i)\rangle$ *equivalence class*." Temporal projection is a mathematic definition of "something" related to time in classical mechanics. Beyond the algebraic aspect, the epistemological interpretation speaks about "the temporal projection of a determined ontic event" or "the formal structure of the duration of a determined ontic event". Let us emphasize that it is explicitly associated to some measurements (localizations) and only implicitly with "a determined ontic event" (only spatially, in this mechanical context reduced to the limit in the present analysis).

Temporal projection thus defined is closely related to a clock and that is it! It is a set of recorded moments, mathematically, it is a subset of **R**. Duration, however, means that we look at the clock. We can simply look at the clock and that is it! But, between two consecutive readings of the clock, be they as precise as they may be, there are unread moments of time about which we can say nothing. A first presupposition is that there are events to which these moments can be associated. The *duration of an event* is a "stronger" presupposition in the sense that between two consecutive, read moments of a determined ontic event all the moments can be associated to the event in question. It is a presupposition! Let us mention that the main points of interest of this analysis have left undiscussed the mathematical aspect that any subset with a cardinal number different from 1 of R is an interval as well as all the consequences (important and interesting but neglected here) that result from this fact. In the formal-algebraic construction which has been proposed, it is not at all necessary for the temporal projection of a determined ontic event, if it is not a number, to be an interval of **R**. In order to discuss duration in the sense of classical mechanics, besides the strong presupposition from above, a measure of the temporal projection must be introduced; in other words, the length of an interval (of time) between two temporal localizations (readings of the clock), that is, an origin of the coordinates, a privileged reference point. Moreover, a consistent epistemological analysis of the uniformity of time can be initiated only if the mechanical notion of duration is completely characterized.

The notion of function was preserved whenever it was necessary, but a "new" concept was introduced into the analysis, the section of a relation according to a subset, which helped avoid some mathematical inconsistencies and allowed for a rigorous algebraic and implicitly epistemologically correct description of a certain class of events. The class of events in question can be interpreted as the duration of an event. This analysis presented the algebraic-epistemological basis of a possible turn to such an interpretation.

Conclusion

This epistemological analysis based on algebraic strictness and abstraction demonstrated among other things how many unclear presuppositions of different nature are introduced, if we may suggestively express ourselves in this way, through a "subconscious intuitive pressure". More simply formulated, our intuition fills automatically the gaps left by a partial theoretical analysis.

In this conclusion we can say something more than at the beginning of the discussion: the clock is an intrinsic component of time, the clock gives *measure* to time; the clock records temporal projections. Now we can understand more precisely a part of the meaning attached to the statement: the clock measures the time. The events do not depend on the time, they only *unfold* in time. By "unfolding in time" we understand that according to a first analysis, events have a determined temporal projection determined by the reading (localization) of a clock.

Epistemologically: the section of an r in a set based on an equivalence class is the formal structure of duration in an algebraic sense. Metaforically: the section of an r in a set based on an equivalence class is a metaphor of the metaphor: it is the "bed" of the "time flow".

Translated by Ágnes Korondi